UNIWERSYTET

KOMISJI EDUKACJI NARODOWEJ

W KRAKOWIE

INSTYTUT MATEMATYKI
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Postulation of lines in the projective space of dimension 3.

Warunki na formy jednorodne zadawane przez proste w trójwymiarowej przestrzeni rzutowej.

praca dyplomowa
promotor: prof. dr hab. Tomasz Szemberg.

Kraków 2025

Za nieoceniony wkład w nasz matematyczny rozwój,
składamy serdeczne podziękowania prof. Tomaszowi Szembergowi,
który prowadził nas zarówno przy realizacji pracy licencjackiej,
jak i wspomagał przez wszystkie lata studiów.

Contents

Chapter 1

Introduction

The purpose of this work is to study the postulation of lines in the projective space of dimension 3. This subject is not new, it has been considered in the 80s by Harshorne and Hirschowitz in [9]. The authors proved there that general lines (when taken with the reduced structure) always impose the expected number of conditions on forms of arbitrary degree in $\mathbb{P}^3$. Our aim here is to present a more detailed proof of this claim and to make the proof available to people with limited command of French.

Postulation problems in geometry have a long and rich history. They have led to many developments in algebraic geometry and commutative algebra. Some problems, even if their formulation can be easily understood, are still widely open.

The story begins with the postulation of points in the projective plane $\mathbb{P}^2$. Given a finite number r of *general* points, it is expected that they impose the expected number of conditions on forms of any degree d. But what is the *expected number of conditions*?

It is well known that the space $H^0(\mathbb{P}^2, \mathcal{O}(d))$ of forms of degree d on $\mathbb{P}^2$ has (affine) dimension $\binom{d+2}{2}$. Vanishing of such a non-zero form in a point imposes one linear condition on the coefficients of the form. If the number of points r satisfies

$$r < \binom{d+2}{2},$$

then we expect that the space of forms vanishing in these points has the dimension

$$\binom{d+2}{2} - r.$$

If the number of points is equal or exceeds the dimension of $H^0(\mathbb{P}^2, \mathcal{O}(d))$, then we expect that there are no such forms apart of the zero form. These expectations are fulfilled by definition for points in the plane.

However changing the setting just a little bit and allowing points with non-reduced structure leads to serious complications. Let us illustrate it by a simple example.

Example 1.1. Let $P, Q \in \mathbb{P}^2$ be two distinct points. Vanishing to order 2 at any of them imposes 3 conditions (all partial derivatives of order 1 at a point must vanish). Since the space $H^0(\mathbb{P}^2, \mathcal{O}(2))$ of forms of degree 2 has dimension 6, we expect that no form of degree 2 exists, which vanish simultaneously at P and Q to order 2. However, the square of the linear form vanishing simply at either point, has this property. So in this case

$$\dim(H^0(\mathbb{P}^2, \mathcal{O}(2) \otimes \mathcal{I}_P \otimes \mathcal{I}_Q) = 6 - 5 = 1.$$

Hartshorne and Hirschowitz instead of increasing the multiplicity of considered points, increased the dimension of the considered base locus and replaced points by general lines. Vanishing along lines in $\mathbb{P}^2$ is easy to detect, because a form vanishes along a line if it is divisible by its equation. The problem becomes interesting in higher dimensional projective spaces. In this work we consider $\mathbb{P}^3$.

The first crucial difference between points in $\mathbb{P}^2$ and lines in $\mathbb{P}^3$ is that the number of conditions imposed by a line on forms of certain degree d depends on the degree (and is equal $d+1$). As a consequence, the number of conditions imposed by a line does not always divide the number of forms of fixed degree. For example the space of forms of degree 5 has affine dimension 56 and the number of conditions imposed by a single line is 6. Since it is convenient to deal with subschemes imposing as many conditions as the dimension of the space of forms, we need to introduce some points in addition.

Our approach is based on the well established specialization and degeneration method exploiting the Castelnuovo Lemma. In order that this approach works we have to deal with some unreduced subschemes.

It is worth to mention that apart of the paper by Hartshorne and Hirschowitz [9] (in French) which exploits degenerations to a smooth quadric in $\mathbb{P}^3$, there is another work by Aladpoosh and Catalisano [1], where the authors consider degenerations to the double plane. A recent preprint by Dumnicki, Malara, Szemberg, Szpond, Tutaj-Gasińska and the first author [5] takes yet another turn and exploits degenerations to a plane.

Chapter 2

Background and Preliminaries

In this chapter we collect initial data which is used in the sequel and is not explained in the standard courses in the mathematics study. We work over the field of complex numbers $\mathbb{C}$.

Let us begin with the ambient space, where our considerations take place, i.e., with the complex projective space.

Definition 2.1. A complex projective space $\mathbb{P}^3$ is defined as the set of equivalence classes of the points in $\mathbb{C}^4 - \{(0, \ldots, 0)\}$ under the equivalence relation

$$(x_0, \ldots, x_3) \sim (\lambda x_0, \ldots, \lambda x_3), \lambda \in \mathbb{C} - \{0\}.$$

Definition 2.2. Let G be an additively written commutative monoid. By a G-graded ring, we mean a ring R, that as an additive group can be expressed as a direct sum.

$$R = \bigoplus_{d \in G} R_d.$$

The elements of R_d are called homogeneous elements of degree d.

Definition 2.3. An ideal I of a graded ring R is homogeneous, if I is generated by homogeneous elements.

Definition 2.4. Let $k[x_0, \ldots, x_n]$ be a ring of homogeneous polynomials, and let I be a proper ideal. Then, I is prime if $fg \in I$ implies $f \in I$ or $g \in I$.

Definition 2.5. Let $\mathbb{C}[x_0, \ldots, x_n]$ be a ring of homogeneous polynomials. The projective algebraic set defined by a homogeneous ideal $I \subset \mathbb{C}[x_0, \ldots, x_n]$ is:

$$V(I) = \{(a_0 : \ldots : a_n) \in \mathbb{P}^n : f(a_0, \ldots, a_n) = 0 \text{ for all } f \in I\}.$$

The ideal of projective algebraic set V, is defined as:

$$I(V) = \{f \in \mathbb{C}[x_0, \ldots, x_n] : f(a_0, \ldots, a_n) = 0 \text{ for all } (a_0 : \ldots : a_n) \in V\}.$$

Note that $I(V)$ is always a saturated ideal.

Definition 2.6. For a commutative ring R with unity, the spectrum of R, denoted $Spec(R)$, is the set of all prime ideals of R.

On $Spec(R)$, we define the Zariski topology. For any ideal $I \subseteq R$, a closed set $V(I)$ is given by:

$$V(I) = \{\mathfrak{p} \in Spec(R) | I \subseteq \mathfrak{p}\}$$

Next, we define the structure sheaf $\mathcal{O}_{Spec(R)}$ on $Spec(R)$.

Definition 2.7. For any open set $U \subseteq Spec(R)$, $\mathcal{O}_{Spec(R)}(U)$ cosists of functions $\rho : U \to \bigcup_{\mathfrak{p} \in U} R_{\mathfrak{p}}$ such that:

- $s(\mathfrak{p}) \in R_{\mathfrak{p}}$ for every $\mathfrak{p} \in U$.
- For every $\mathfrak{p} \in U$, there exist an open neighborhood $V \subseteq U$ of $\mathfrak{p}$ and elements $g, f \in R$ such that $f \notin \mathfrak{q}$ for all $\mathfrak{q} \in V$, and $s(\mathfrak{q}) = \frac{g}{f}$ in $R_{\mathfrak{q}}$ for all $\mathfrak{q} \in V$.

The pair $(Spec(R), \mathcal{O}_{Spec(R)})$ is called an affine scheme.

Definition 2.8. A scheme is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighborhood U such that the topological space U, together with the restricted sheaf $\mathcal{O}_X\big|_U$, is an affine scheme. We call X the underlying topological space of the scheme $(X, \mathcal{O}_X)$, and $\mathcal{O}_X$ its structure sheaf.

Theorem 2.9. *A projective algebraic set V is irreducible if and only if $I(V)$ is a prime ideal.*

Now we recall one of the fundamental results in algebraic geometry which is Bézout's Theorem.

Theorem 2.10. *If F and G are curves in $\mathbb{P}^2$, which are defined by homogeneous polynomials, of degree d_1, d_2, respectively, and do not have any common components, then the sum of the intersection multiplicities at all their intersection points is equal to $d_1 d_2$:*

$$\sum_{P \in F \cap G} I(P, F \cap G) = d_1 d_2.$$

where $I(P, F \cap G)$ denotes the intersection multiplicity of F and G at the point P.

Example 2.11. Let F be curve of degree 3 and G a curve of degree 2 in $\mathbb{P}^2$, then the total number of intersection points (counted with multiplicities) is $3 \cdot 2 = 6$

A powerful generalization of Bézout's Theorem concerns the intersection of algebraic sets with hypersurfaces.

Let $V \subset \mathbb{P}^n$ be a projective algebraic set of dimension δ and degree d_1 defined by a homogeneous ideal $I \subset \mathbb{C}[x_0, \ldots, x_n]$, and let $H \subset \mathbb{P}^n$ be a hypersurface defined by a homogeneous polynomial of degree d_2, such that H does not contain any irreducible component of V. Then the intersection $V \cap H$ is a subscheme of dimension $\delta - 1$, and its degree satisfies:

$$deg(V \cap H) = deg(V) \cdot deg(H) = d_1 \cdot d_2.$$

This version of Bézout's Theorem is an important computational tool in the intersection theory.

2.1 Conditions in $\mathbb{P}^n$ imposed by vanishing along lines

Theorem 2.12. *A single line L imposes $d+1$ conditions on the forms of degree d in $\mathbb{P}^3$.*

Proof. All lines in $\mathbb{P}^3$ are projectively equivalent. Since number of imposed conditions is invariant under linear change of coordinates, it suffices to prove the theorem for a conveniently chosen line L.

Let $L = \{(x : y : z : w) \in \mathbb{P}^3 : x = y = 0\}$. We first show the claim for small values of d. Let us begin with forms of degree $d = 1$.

In order to parametrize L, we choose two distinct points $P = (0 : 0 : 1 : 0)$, $P' = (0 : 0 : 0 : 1)$ on L. Then

$$\psi : \mathbb{P}^1 \to L, \quad (\lambda, \mu) \mapsto \lambda P + \mu P' = (0 : 0 : \lambda : \mu)$$

parametrizes L. Using this parametrization, we claim that L is contained in a form of degree 1 defined by the equation $Ax+By+Cz+Dw = 0$ if and only if for all $(\lambda : \mu) \in \mathbb{P}^1$

$$A \cdot 0 + B \cdot 0 + C \cdot \lambda + D \cdot \mu = 0.$$

This is equivalent to

$$\lambda C + \mu D = 0$$

All coefficients of a zero polynomial are equal to zero, thus it must be

$$\begin{cases} C &= 0 \\ D &= 0 \end{cases}.$$

The codimension of the subspace of all forms of degree 1 vanishing on L is equal to 2, this follows from the above linear system of equations with rank 2 coefficient matrix (simply, those equations are linearly independent). By the definition, this is equivalent to the fact, that L imposes 2 conditions on forms of degree 1, which proves the statement for degree 1 and an arbitrary line.

The proof for the forms of degree $d = 2$ is done by the same strategy. The line L, points P, P' and the parametrization remain unchanged. The only difference is that now we consider a quadric Q defined by the equation

$$Ax^2 + By^2 + Cz^2 + Dw^2 + Exy + Fxz + Gxw + Hyz + Iyw + Jzw = 0.$$

For a line L we have inclusion $L \subset Q$ if and only if for all $(\lambda : \mu) \in \mathbb{P}^1$

$$A \cdot 0^2 + B \cdot 0^2 + C \cdot \lambda^2 + D \cdot \mu^2 + E \cdot 0 + F \cdot 0 \cdot \lambda + G \cdot 0 \cdot \mu \ + H \cdot 0 \cdot \lambda + I \cdot 0 \cdot \mu + J \cdot \lambda \cdot \mu = 0$$

which gives

$$\lambda^2 C + \mu^2 D + \lambda\mu J = 0$$

this implies

$$\begin{cases} C &= 0 \\ D &= 0 \\ J &= 0 \end{cases}.$$

The rank of coefficient matrix of above system of linear equation is equal to 3, therefore the codimension of the subspace of the forms of degree 2 vanishing on L in the space of all forms of degree 2 is equal to 3, which is equivalent to the fact, that every line in $\mathbb{P}^3$ implies 3 conditions on forms of degree 2.

Now we consider the general case of forms of degree d. The line L, points P, P' and line parametrization remain unchanged. Let F be a form of degree d. Every such a form is defined by equation:

$$F(x,y,z,w) = \sum_{i+j+k+l=d} c_{ijkl} x^i y^j z^k w^l.$$

Notice that the number of monomials of degree d in $\mathbb{P}^3$ is equal to $\binom{3+d}{d}$. The form F vanishes along the line L if and only if for all $(\lambda : \mu) \in \mathbb{P}^1$

$$F(0,0,\lambda,\mu) = \sum_{i+j+k+l=d} c_{ijkl} \cdot 0^i \cdot 0^j \cdot \lambda^k \cdot \mu^l = 0.$$

Notice that the only non-zero components are those where $i = j = 0$. There are $\binom{1+d}{d} = d+1$ such components. Therefore the system of linear equation is given by $d+1$ equations. Every equation has the form

$$c_{00jk} = 0.$$

The coefficient matrix has clearly the maximal rank equal to $d+1$, since every equation refers to a different variable. This is equivalent to the fact, that the line L imposes $d+1$ conditions on forms of degree d. □

The Theorem 2.12 can be easily generalized for projective spaces of any dimension.

Theorem 2.13. *A single line L imposes $d+1$ conditions on the forms of degree d in $\mathbb{P}^n$ for any $n \geqslant 1$.*

Proof. The proof can be done using similar methods to the previous Theorem 2.12. We start with stating the fact, that every line in $\mathbb{P}^n$ is equivalent up to the projective change of variables, which does not affect the number of implied conditions on any forms. Therefore it suffices to prove the statement for a conveniently chosen line L.

Let $L = \{(x_0 : x_1 : \ldots : x_n) \in \mathbb{P}^n : x_0 = x_1 = \ldots = x_{n-2} = 0\}$. In order to parametrize L we choose two distinct points $P = (0 : \ldots : 0 : 1 : 0)$, $P' = (0 : \ldots : 0 : 0 : 1)$ on L. Then

$$\psi : \mathbb{P}^1 \to L, \quad (\lambda, \mu) \mapsto \lambda P + \mu P' = (0 : \ldots : 0 : \lambda : \mu)$$

parametrizes L. Consider general form of degree d given by an equation:

$$F(x_0, \ldots, x_n) = \sum_{i_0+\ldots+i_n=d} c_{i_0 \ldots i_n} \cdot x_0^{i_0} \cdot \ldots \cdot x_n^{i_n} = 0.$$

$L \subset F$ if and only if for all $(\lambda : \mu) \in \mathbb{P}^1$

$$F(0, \ldots, 0, \lambda, \mu) = \sum_{i_0+\ldots+i_n=d} c_{i_0 \ldots i_n} \cdot 0^{i_0} \cdot \ldots \cdot 0^{i_{n-2}} \cdot \lambda^{i_{n-1}} \cdot \mu^{i_n} = 0$$

Notice that, all nonzero components of above sum have $i_0 = \ldots = i_{n-2} = 0$. These components therefore fulfill the equation $i_{n-1} + i_n = d$. There are $\binom{1+d}{d} = d+1$ such components. Every one of them gives an equation

$$c_{0\ldots0 i_{n-1} i_n} = 0$$

Every equation is linearly independent, therefore the codimension of a subspace of all forms of degree d vanishing on line L is equal to $d+1$, which proves the Theorem. $\square$

The result above leads to a question about points on a line. In particular, we can ask whether a finite set of collinear points imposes independent conditions on homogeneous forms of a given degree. The following lemma addresses this question in the case of $\mathbb{P}^3$.

Lemma 2.14. *Let $p_1, \ldots, p_q$ be collinear points in $\mathbb{P}^3$. If $q \leqslant k+1$, then these points impose independent conditions on forms of degree k.*

Proof. For any $q \leqslant k+1$, we can construct a hypersurface of degree k vanishing at any chosen subset of k of the points, but not at the remaining one. Indeed, for each point p_i, we can choose a hyperplane containing only p_i among the selected points. Taking the union of such hyperplanes gives a hypersurface of degree k that vanishes at exactly k chosen points. Therefore, the conditions imposed by the points are linearly independent. If $q > k+1$, then every form od degree k that vanishes at $k+1$ collinear points, must also vanish on the entire line, and hence at all q points. In this case, the conditions are dependent, and the number of independent ones is bounded above by $k+1$. $\square$

Chapter 3

The multiprojective space $\mathbb{P}^1 \times \mathbb{P}^1$ and its relation to quadrics in $\mathbb{P}^3$.

In this chapter we relate via the Segre map the product $\mathbb{P}^1 \times \mathbb{P}^1$ and a smooth quadric in $\mathbb{P}^3$.

3.1 Introductory remarks on multiprojective spaces

Definition 3.1. The Segre embedding of $\mathbb{P}^n \times \mathbb{P}^m$ is the map

$$S : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{(n+1)(m+1)-1}$$

defined pointwise as follows:

$$S((x_0 : x_1 : \ldots : x_n), (y_0 : y_1 : \ldots : y_m)) = (x_0y_0 : x_0y_1 : \ldots : x_ny_m).$$

In the most basic case, namely when $n = m = 1$ this gives an embedding of the product of the projective line with itself into $\mathbb{P}^3$. The image of this embedding is a smooth quadric surface, which clearly contains two one-parameter families of lines. Over the complex numbers, any smooth quadric is ruled in two distinct ways by lines, so that any smooth quadric in $\mathbb{P}^3$ can be parametrized by the Segre map.
From now on let $S : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ denote the map given by

$$S((u : v), (s, t)) = (us : ut : vs : vt). \tag{3.1}$$

Definition 3.2. Let $F(x, y)$ be a polynomial in $(x_0, x_1, \ldots, x_m)$ and $(y_0, y_1, \ldots, y_n)$. We say that $F(x, y)$ is bihomogeneous of bi-degree (a, b) if it is homogeneous of degree a in the variables $(x_0, x_1, \ldots, x_m)$ and of degree b in the variables $(y_0, y_1, \ldots, y_n)$.

Example 3.3. The polynomial

$$F(s : t, u : v) = su^2 + vut - v^2t$$

is bihomogeneous of bi-degree $(1, 2)$. Note that the zero locus of a bihomogeneous polynomial is a well-defined subset of the product of projective spaces.

Lemma 3.4. *The dimension of the space of bihomogeneous polynomials of bidegree* (a, b) *in* $\mathbb{P}^m \times \mathbb{P}^n$ *is equal to* $\binom{m+a}{a}\binom{n+b}{b}$.

Proof. Let's take a bihomogeneous polynomial of degree (a, b) in variables $x = (x_0, \ldots, x_m)$ and $y = (y_0, \ldots, y_n)$. The dimension of the space of bihomogeneous polynomials of bidegree (a, b) corresponds to the number of different monomials of this degree. The number of different monomials of degree a in $x = (x_0, \ldots, x_m)$ is given by $\binom{m+a}{a}$. Similarly, the number of different monomials of degree b in $y = (y_1, \ldots, y_n)$ is given by $\binom{n+b}{b}$. Since each x-monomial of degree a can be combined with each y-monomial of degree b, the total number of bihomogeneous monomials of degree (a, b) is the product of :

$$\binom{m+a}{a}\binom{n+b}{b}$$

For $m = 1$ and $n = 1$ the binomial coefficients simplify to:

$$\binom{1+a}{a}\binom{1+b}{b} = (a+1)(b+1).$$

□

We define $\mathbb{P}^1 \times \mathbb{P}^1$ as the product of two projective lines. Therefore, we write:

$$\mathbb{P}^1 = \mathbb{A}^1 \cup \{(0:1)\}.$$

Every point in the first copy of $\mathbb{P}^1$ corresponds to a vertical line in $\mathbb{P}^1 \times \mathbb{P}^1$, while every point in the second copy gives rise to horizontal line. In particular, the point at infinity $(0 : 1)$ in the first factor correspond to the vertical line:

$$\mathfrak{L}_1 = \{(0:1)\} \times \mathbb{P}^1,$$

and the point at infinity $(0 : 1)$ in the second factor gives the horizontal line:

$$\mathfrak{L}_2 = \mathbb{P}^1 \times \{(0:1)\}.$$

These two lines, $\mathfrak{L}_1$ and $\mathfrak{L}_2$, play the fundamental role in $\mathbb{P}^1 \times \mathbb{P}^1$: any curve on the surface can be expressed as a linear combination of them.

In the multiprojective space $\mathbb{P}^1 \times \mathbb{P}^1$, we can formulate a version of Bézout's theorem analogous to the classical one in $\mathbb{P}^2$.

Theorem 3.5. *Let* C *and* D *be two curves in* $\mathbb{P}^1 \times \mathbb{P}^1$ *without common components. Suppose that* C *has bidegree* (a_1, b_1) *and* D *has bidegree* (a_2, b_2). *Then the number of points in the intersection* $C \cap D$, *counted with multiplicities, is given by*

$$C \cdot\ D = \sum_{P \in C \cap D} (C \cdot D)_P = a_1 b_2 + b_1 a_2.$$

Where, $(C \cdot D)_P$ *denotes the local intersection multiplicity at the point* P.

3.2 The Segre map and the isomorphism of $\mathbb{P}^1 \times \mathbb{P}^1$ with smooth quadrics in $\mathbb{P}^3$

Now we will show that $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to a smooth quadric in $\mathbb{P}^3$. We endow the first factor in the product $\mathbb{P}^1 \times \mathbb{P}^1$ with coordinates $(u : v)$ and the second factor with $(s : t)$.

Consider the following morphism (the Segre map):

$$S : \mathbb{P}^1 \times \mathbb{P}^1 \ni ((u : v), (s : t)) \to (us : ut : vs : vt) \in \mathbb{P}^3.$$

Let $(x_0 : x_1 : x_2 : x_3)$ be coordinates on $\mathbb{P}^3$. In these coordinates the image of S satisfies the following equation:

$$x_0 x_3 - x_1 x_2 = 0, \tag{3.2}$$

which defines a smooth quadric Q in $\mathbb{P}^3$.

In the other direction, let $P = (p_0 : p_1 : p_2 : p_3)$ be a point in Q, i.e., $p_0 p_3 = p_1 p_2$.

For $p_1 \neq 0$ we define the map $F_1 : (p_0 : p_1 : p_2 : p_3) \to ((p_1 : p_3), (p_0 : p_1))$. Using (3.1), we obtain

$$S((p_1 : p_3), (p_0 : p_1)) = (p_0 p_1 : p_1^2 : p_0 p_3 : p_1 p_3).$$

Since, $p_0 p_3 = p_1 p_2$ we have the following equality:

$$(p_0 p_1 : p_1^2 : p_0 p_3 : p_1 p_3) = (p_0 p_1 : p_1^2 : p_1 p_2 : p_1 p_3).$$

Dividing each coordinate by p_1, we get

$$(p_0 p_1 : p_1^2 : p_0 p_3 : p_1 p_3) = (p_0 : p_1 : p_2 : p_3),$$

which means that F_1 inverses S on $U_1 = \{(p_1 : p_2 : p_3 : p_4) : \ p_1 \neq 0\} \subset Q$.

For $p_2 \neq 0$ we define the map $F_2 : (p_0 : p_1 : p_2 : p_3) \to ((p_0 : p_2), (p_2 : p_3))$. Similarly to the previous case, we have

$$(p_0 p_2 : p_0 p_3 : p_2^2 : p_2 p_3) = (p_0 p_2 : p_1 p_2 : p_2^2 : p_2 p_3).$$

Dividing each coordinate by p_2, we obtain

$$(p_0 p_2 : p_1 p_2 : p_2^2 : p_2 p_3) = (p_0 : p_1 : p_2 : p_3),$$

which means that F_2 inverses S on $U_2 = \{(p_1 : p_2 : p_3 : p_4) : \ p_2 \neq 0\} \subset Q$.

The cases for nonzero p_3 and p_4, are analogous.

In the final step we need to show that the maps F_1, F_2, F_3 and F_4 agree on the intersection of their domains, providing a global morphism inverting Q. For example for

$$P = (p_0 : p_1 : p_2 : p_3) \in U_1 \cap U_2$$

we have

$$F_1(P) = ((p_1 : p_3), (p_0 : p_1)) \in \mathbb{P}^1 \times \mathbb{P}^1$$

and

$$F_2(P) = ((p_0 : p_2), (p_2 : p_3)) \in \mathbb{P}^1 \times \mathbb{P}^1.$$

We claim that

$$((p_1 : p_3), (p_0 : p_1)) = ((p_0 : p_2), (p_2 : p_3)).$$

It suffices to check that

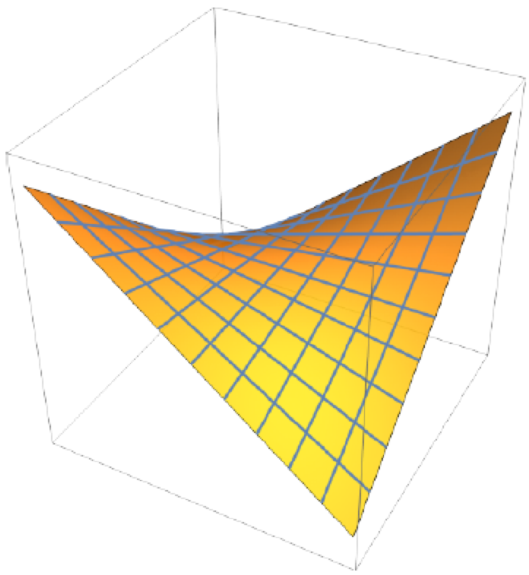
$$(p_1 : p_3) = (p_0 : p_2) \quad \text{and} \quad (p_0 : p_1) = (p_2 : p_3).$$

Since $P \in Q$ it must be $p_0 p_3 = p_1 p_2$ and the claim follows as

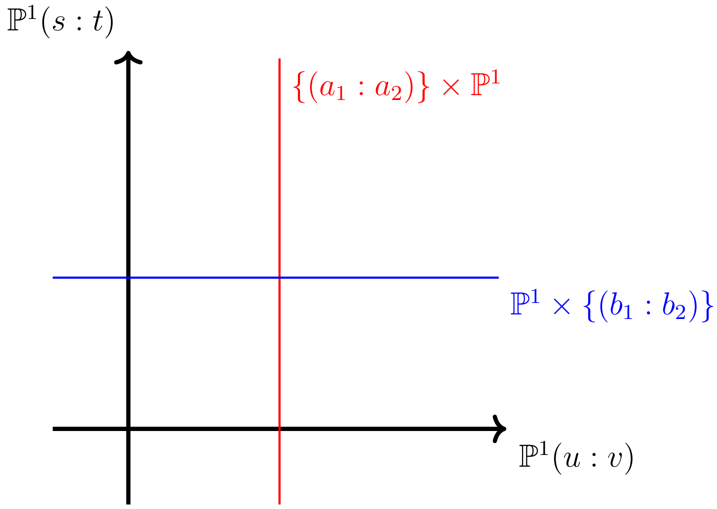
$$\det \begin{pmatrix} p_0 & p_2 \\ p_1 & p_3 \end{pmatrix} = \det \begin{pmatrix} p_2 & p_3 \\ p_0 & p_1 \end{pmatrix} = 0.$$

As we saw above the projective surface $\mathbb{P}^1 \times \mathbb{P}^1$ can be embedded in $\mathbb{P}^3$ as $x_0 x_3 - x_1 x_2 = 0$. Restricting to the affine chart where $x_0 \neq 0$, this surface is given by the equation $x_3 = x_1 x_2$ in $\mathbb{A}^3$. Over $\mathbb{R}$, this surface is double ruled, meaning that it is covered by a family of disjoin lines in two distinct ways.

Figure 3.1: Surface given by the equation $x_3 = x_1 x_2$ in $\mathbb{R}^3$. Source: [3]



The affine picture extends to the projective setting, namely the projective surface $\mathbb{P}^1 \times \mathbb{P}^1$ itself is double ruled in $\mathbb{P}^3$, as is contains two families of lines corresponding to the sets of form $\{(a_1 : a_2)\} \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \{(b_1 : b_2)\}$.



3.3 Preimage under the Segre map of curves on a smooth quadric in $\mathbb{P}^3$

In the proof of the main Theorem 4.1, we will strongly use the isomorphism between smooth quadric on $\mathbb{P}^3$ and $\mathbb{P}^1 \times \mathbb{P}^1$ given by Segre morphism. However, to complete our proofs, we need to understand how we can view curves embedded in quadrics in $\mathbb{P}^3$ as curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

From now on, every curve or a point on $\mathbb{P}^1 \times \mathbb{P}^1$ we denote using Fraktur letters e.g. $\mathfrak{C}, \mathfrak{L}, \mathfrak{p}$ to differentiate them from the curves and points on $\mathbb{P}^3$ denoted by the same letters in the standard font.

Consider a smooth quadric $Q \subset \mathbb{P}^3$ given by the equation $x_0x_3 - x_1x_2 = 0$ and an irreducible surface $F \subset \mathbb{P}^3$ given by the equation $f = 0$ for some homogeneous polynomial f of degree d in variables x_0, x_1, x_2, x_3. If $Q \not\subset F$, then from Bézout's Theorem 2.10 we know that $C = F \cap Q$ (the intersection taken scheme theoretic) is a subscheme of degree $2d$ embedded in Q.

Theorem 3.6. *Let Q, F be defined as above. Let $S : \mathbb{P}^1 \times \mathbb{P}^1 \to Q$ denote the Segre isomorphism defined earlier. Then $\mathfrak{C} = S^{-1}(F \cap Q)$ is a curve of bidegree (d, d) in $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Consider the point $((u_0 : v_0), (s_0 : v_0) \in S^{-1}(F \cap Q)$. From the definition of preimage we derive

$$\exists (x_0 : x_1 : x_2 : x_3) \in F \cap Q : S((u_0 : v_0), (s_0 : t_0)) = (x_0 : x_1 : x_2 : x_3).$$

Notice that $S((u_0 : v_0), (s_0 : t_0)) = (u_0s_0 : u_0t_0 : v_0s_0 : v_0t_0)$ and therefore $S((u_0 : v_0), (s_0 : t_0)) \in Q$ is satisfied regardless of u_0, v_0, s_0, t_0. It remains to show the condition under which $(u_0s_0 : u_0t_0 : v_0s_0 : v_0t_0) \in F$. Naturally, the point lies in a surface if and only if it satisfies it's equation. Thus we have

$$((u_0 : v_0), (s_0 : t_0)) \in S^{-1}(F \cap Q) \Leftrightarrow f(u_0s_0 : u_0t_0 : v_0s_0 : v_0t_0) = 0.$$

Since $f(x_0 : x_1 : x_2 : x_3)$ was a polynomial of degree d it is easy to see that $f(u_0s_0 : u_0t_0 : v_0s_0 : v_0t_0)$ is a bihomogeneous polynomial of bidegree (d, d) and it defines a curve of such bidegree in $\mathbb{P}^1 \times \mathbb{P}^1$. □

Notice that Theorem 3.6 does not give us the information on the Segre morphism preimage of a line embedded in a quadric in $\mathbb{P}^3$, since a single line (with reduced structure) cannot be obtained as the intersection of some surface F with the smooth quadric Q. However, notice that if we take a point $P \in Q$, then the intersection of the plane H tangent to Q in point P with quadric Q will be a curve of degree 2 degenerated to two lines intersecting exactly in the point P. This construction will be useful in the proof of the statement that the preimage of the Segre morphism of a line contained in Q is, in fact, a line in $\mathbb{P}^1 \times \mathbb{P}^1$. But first we need to prove the following lemma.

Lemma 3.7. *Let $\mathfrak{L} \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a line, Q the smooth quadric defined earlier, and $S : \mathbb{P}^1 \times \mathbb{P}^1 \to Q$ the Segre isomorphism defined earlier. Then $S(\mathfrak{L})$ is a line in $\mathbb{P}^3$ contained in Q.*

Proof. Since $\mathfrak{L}$ is a line in $\mathbb{P}^1 \times \mathbb{P}^1$, it is given by the zeros of a bihomogeneous polynomial g of bidegree $(1, 0)$ or bidegree $(0, 1)$. Without loss of generality, we may assume that g is of bidegree $(1, 0)$. Thus, we can write $g((u : v),(s : t)) = au + bv$ for some $a, b \in \mathbb{C}$ not vanishing simultaneously. From this formula, we obtain the parametrization of $\mathfrak{L}$ given by $\mathfrak{L} = \{\ ((b : -a),(s : t)) \mid (s : t) \in \mathbb{P}^1\}$.

Hence

$$S(\mathfrak{L}) = \{(bs : bt : -as : -at) \mid (s : t) \in \mathbb{P}^1\}.$$

From this parametrization we can read of generators of the ideal of $S(\mathfrak{L})$:

$$\begin{cases} ax_0 + bx_2 \\ ax_1 + bx_3 \end{cases}.$$

Thus, $S(\mathfrak{L})$ is a line in $\mathbb{P}^3$ lying on a quadric Q. □

This gives us the tools required to prove the following important statement.

Theorem 3.8. *Let L be a line in $\mathbb{P}^3$ contained in Q, which is a quadric defined earlier. Let S denote the Segre isomorphism defined earlier. Then $S^{-1}(L)$ is a line in $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Let $g = x_0x_3 - x_1x_2$ be the equation of Q.

We begin by choosing any point $p \in L$. Since $L \subset Q \cong \mathbb{P}^1 \times \mathbb{P}^1$, the point p can be expressed as the image $p = S((u_0 : v_0),(s_0 : t_0)) = (u_0s_0 : u_0t_0 : v_0s_0 : v_0t_0)$ for some $((u_0 : v_0),(s_0 : t_0)) \in \mathbb{P}^1 \times \mathbb{P}^1$.

Let H be a plane tangent to Q at the point p. We know that H is given by the equation $f = 0$, where $f = \sum_{i=0}^{3} \dfrac{\partial g}{\partial x_i} p x_i$. By calculating the partial derivative, we obtain $f = v_0t_0x_0 - v_0s_0x_1 - u_0t_0x_2 + u_0s_0x_3$. The crucial observation mentioned earlier is that $H \cap Q = L \cup L'$ where L' is another line contained in Q that intersects with L exactly at p. Now consider the point $\mathfrak{p} = ((u : v),(s : t)) \in \mathbb{P}^1 \times \mathbb{P}^1$ such that $\mathfrak{p} \in S^{-1}(H \cap Q)$. From the proof of Theorem 3.6, we know that $\mathfrak{p} \in S^{-1}(H \cap Q) \Leftrightarrow f(S(\mathfrak{p})) = 0$. Substituting, we obtain

$$0 = f(S(\mathfrak{p})) = v_0t_0us - v_0s_0ut - u_0t_0vs + u_0s_0vt = (u_0v - v_0s)(s_0t - t_0s).$$

Therefore, $S^{-1}(H \cap Q) = \mathfrak{L}_1 \cup \mathfrak{L}_2$, where $\mathfrak{L}_1, \mathfrak{L}_2$ are lines in $\mathbb{P}^1 \times \mathbb{P}^1$ given by equations $u_0v - v_0s = 0$ and $s_0t - t_0s = 0$ respectively. We have

$$\mathfrak{L}_1 \subset S^{-1}(H \cap Q) \Rightarrow S(\mathfrak{L}_1) \subset S(S^{-1}(H \cap Q)) = H \cap Q = L \cup L'.$$

By Lemma 3.7 we know that $S(\mathfrak{L}_1)$ is a line in $\mathbb{P}^3$ therefore it must be either L or L'. Without loss of generality, we may assume that $S(\mathfrak{L}_1) = L$. From this it is easy to see that $S(\mathfrak{L}_2) = L'$. From the fact that S is injective, we know that $S^{-1}(L) = \mathfrak{L}_1$, therefore, it ends the proof. □

By Lemma 3.7 and Theorems 3.6 and 3.8 we get a good understanding of how curves contained in the quadrics in $\mathbb{P}^3$ correspond to curves on $\mathbb{P}^1 \times \mathbb{P}^1$. We conclude this section with somewhat obvious corollaries. As $\mathbb{P}^1 \times \mathbb{P}^1$ there are two families of lines on a smooth quadric on $\mathbb{P}^3$, they are called *rulings*. Lines of the same family do not intersect, and any two lines of distinct rulings intersect at exactly one point in Q. If we take the inverse of Segre embedding, the lines of the same ruling in $\mathbb{P}^3$ lie on the same ruling in $\mathbb{P}^1 \times \mathbb{P}^1$ and the lines of distinct rulings in $\mathbb{P}^3$ correspond to the lines of distinct rulings in $\mathbb{P}^1 \times \mathbb{P}^1$.

3.4 Independent conditions imposed by points in $\mathbb{P}^1 \times \mathbb{P}^1$.

In order to prove the main Theorem 4.1, we need to understand conditions imposed by points in $\mathbb{P}^1 \times \mathbb{P}^1$ on forms of all bidegree. We address this problem in this section. Since it is fully dedicated to $\mathbb{P}^1 \times \mathbb{P}^1$, we omit using Fraktur letters, and convey to the standard notation.

Let us introduce some additional notation used throughout this section. Let X be any set. By $\mathcal{F}((a,b), X)$ we denote the space of all forms of bidegree (a,b) vanishing on X. In this notation $\mathcal{F}((a,b), \emptyset)$ is just the space of all forms of bidegree (a,b). For any set X we view $\mathcal{F}((a,b), X)$ as a linear subspace of $\mathcal{F}((a,b), \emptyset)$ and we denote that by $\mathcal{F}((a,b), X) \leqslant \mathcal{F}((a,b), \emptyset)$.

Theorem 3.9. *Let X be the set of k points in general position in $\mathbb{P}^1 \times \mathbb{P}^1$. Then* $\operatorname{codim} \mathcal{F}((a,b), X) = \max\{k, (a+1)(b+1)\}$.

We omit the proof of this theorem. It just states that k points in general position impose k independent conditions on forms of bidegree (a,b). This is the expected behavior, similar to that of general points in $\mathbb{P}^n$.

Much less intuitive is what we state next. If we take k collinear points in $\mathbb{P}^1 \times \mathbb{P}^1$, then on the forms of sufficiently high bidegree they impose independent conditions. This fact is commonly used throughout the proof of the main theorem. To tackle the proof of this, we start with a lemma.

Lemma 3.10. *Let Z be the set of $k+1$ collinear points lying in the vertical ruling of $\mathbb{P}^1 \times \mathbb{P}^1$. Then for any $l \in \mathbb{N}$, Z imposes independent conditions on the forms of bidegree (l,k), that is,* $\operatorname{codim} \mathcal{F}((l,k),Z) = k+1$.

Proof. $\operatorname{codim} \mathcal{F}((l,k),Z) = \dim \mathcal{F}((l,k)\emptyset) - \dim \mathcal{F}((l,k)Z)$. By 3.4, $\dim \mathcal{F}((l,k)\emptyset) = (l+1)(k+1)$. The fact that the points of Z lie in a vertical ruling means that there exists a vertical line, that is, a form of bidegree $(1,0)$, passing through all the points of Z. If the form of bidegree (l,k) vanishes on Z it must contain this line. Thus, we are left with any form of bidegree $(l-1,k)$. Therefore, $\dim \mathcal{F}((l,k)Z) = \dim \mathcal{F}((l-1,k)\emptyset) = l(k+1)$. Thus, $\operatorname{codim} \mathcal{F}((l,k),Z) = (l+1)(k+1) - l(k+1) = k+1$, which ends the proof. □

Now, if a set imposes independent conditions on the forms of some bidegree, we expect it to impose independent conditions on all the forms of "higher" bidegree. We formalize this statement in the following theorem.

Theorem 3.11. *Let Z be the set of $k+1$ collinear points lying in the vertical ruling of $\mathbb{P}^1 \times \mathbb{P}^1$. Then for any $l,u \in \mathbb{N}$ the set Z imposes independent conditions on forms of bidegree $(l,k+u)$, that is,* $\operatorname{codim} \mathcal{F}((l,k+u),Z) = k+1$.

Proof. Let $W = Z + p_{k+2} + \ldots + p_{k+u+1}$ be the union of Z with u additional points on the same vertical line that containing Z. Thus, W is the set of $k+u$ collinear points lying on a vertical line.

Directly from Lemma 3.10 we know that $\operatorname{codim} \mathcal{F}((l,k+u),W) = k+u+1$. Now notice that since $Z \subset W$ we have

$$\mathcal{F}((l,k+u),W) \leqslant \mathcal{F}((l,k+u),Z) \leqslant \mathcal{F}((l,k+u),\emptyset)$$

that is, we can view $\mathcal{F}((l,k+u),W)$ not only as a linear subspace of $\mathcal{F}((l,k+u),\emptyset)$ but also as a linear subspace of $\mathcal{F}((l,k+u),Z)$. The last observation is that the codimension of $\mathcal{F}((l,k+u),W)$ viewed as a subspace of $\mathcal{F}((l,k+u),Z)$ is at most u, since $|W \setminus Z| = u$, and similarly codimension of $\mathcal{F}((l,k+u),Z)$ viewed as a subspace of all forms of bidegree (a,b) is at most $k+1$, since Z consists of $k+1$ points. To acknowledge that it is sufficient to state that $\operatorname{codim} \mathcal{F}((l,k+u),Z) = k+1$, see Figure 3.2. □

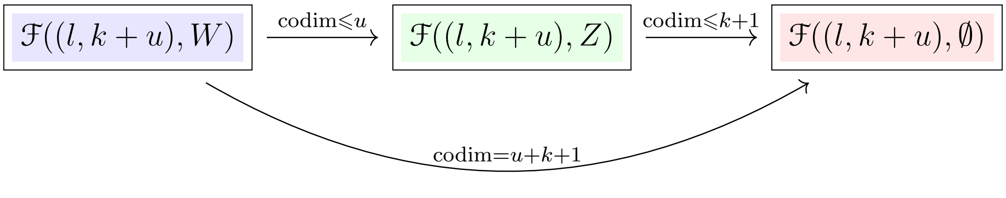


Figure 3.2: A diagram of codimension relations between spaces.

Lemma 3.10 and Theorem 3.11 can naturally be restated and proved for the points in the horizontal ruling. We omit that.

We end this section with the theorem on the independent conditions imposed by configurations of points and double points on $\mathbb{P}^1 \times \mathbb{P}^1$. It is well known that on $\mathbb{P}^2$ a double point (point with multiplicity two) imposes 3 conditions. The same holds for $\mathbb{P}^1 \times \mathbb{P}^1$. However, it is less obvious that if we take, e.g., four points with multiplicity one and two double points, they induce the expected number, i.e., $4 \cdot 1 + 2 \cdot 3 = 10$ conditions on the forms of bidegree $(1, 4)$. To manage such situations, we introduce the following theorem, which is a direct corollary of [12, Theorem 2.2].

Theorem 3.12. *The configuration of $3k^2 - k + 2$ general points with multiplicity one and $2k$ general points with multiplicity two imposes independent conditions on a form of bidegree $(k, 3k + 1)$.*

Chapter 4

The main theorem

In this chapter we state our main result.

Theorem 4.1 (Main theorem). *Let $Y \subset \mathbb{P}^3$ be a set of r lines $L_1, \ldots, L_r$ in general position, $Y = L_1 + \ldots + L_r$. Then Y imposes*

$$\min\left(r(d+1), \binom{3+d}{3}\right)$$

independent conditions on forms of degree d, i.e., the space of forms of degree d vanishing on Y is either zero, or its codimension in the space of all forms of degree d is equal to $r(d+1)$.

An equivalent statement is that the linear transformation (the restriction map)

$$\phi : H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d)) \to H^0(Y, \mathcal{O}_Y(d))$$

derived from the structure sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Y \to 0$$

twisted by d

$$0 \to \mathcal{I}_Y(d) \to \mathcal{O}_{\mathbb{P}^3}(d) \to \mathcal{O}_Y(d) \to 0$$

has always the maximal rank.

Example 4.2. For $d = 3$ the dimension of the space of forms of degree 3 in $\mathbb{P}^3$ is equal to $\binom{3+3}{3} = 20$. We expect for $Y = L_1$ that $\dim\ker\phi = 20 - (3+1)1 = 16$ (so the codimension is equal to 4). For $Y = L_1 + L_2$ the expected $\dim\ker\phi = 20 - (3+1)2 = 12$ and so on. That means that for $Y = L_1 + \ldots + L_5$ the dimension $\dim\ker\phi$ equals 0, i.e. the kernel is trivial, so the only form of degree 3 vanishing on 5 general lines is the zero polynomial. The maximal rank statement can be understood by the fact that for $r < 5$ ϕ is expected to be surjective, for $r = 5$ bijective and for all $r > 5$ injective.

This example leads to the following generalization. If $r_0 = \dfrac{\binom{3+d}{d}}{d+1}$ is an integer, then for $Y = L_1 + \ldots + L_{r_0}$ the map ϕ in Theorem 4.1 is expected to be bijective. Once proved, this claim implies the Main Theorem for any number r of lines. Indeed, the restriction map must be injective (having the trivial kernel) for all $r > r_0$. For $r < r_0$, let $Y_r = L_1 + \ldots + L_r$ and $Y_{r_0} = Y_r + L_{r_0+1} + \ldots + L_{r_0}$. We have $Y_r \subset Y_{r_0}$ and

$$H^0(Y_r, \mathcal{O}_{Y_r}(d)) = \oplus_{i=1}^{r} H^0(L_i, \mathcal{O}_{Y_i}(d)) \subset \oplus_{i=1}^{r_0} H^0(L_i, \mathcal{O}_{Y_i}(d)) = H^0(Y_{r_0}, \mathcal{O}_{Y_{r_0}}(d)).$$

Hence there is a natural projection on a subspace

$$\pi : H^0(Y_{r_0}, \mathcal{O}_{Y_{r_0}}(d)) \to H^0(Y_r, \mathcal{O}_{Y_r}(d))$$

and $\phi_r = \phi_{r_0} \circ \pi$ is a surjection.

The number r_0 is an integer provided that d is sufficiently divisible. Additional difficulties emerge when this condition is not satisfied.

Example 4.3. Consider $d = 5$. Let $\mathbb{P}^3 \supset Y = L_1 + \cdots L_9$ be 9 lines in general position and similarly $\mathbb{P}^3 \supset Y' = L'_1 + \ldots + L'_{10}$ be 10 lines in general position.

By Lemma 3.4 the dimension of the space of forms of degree 5 in $\mathbb{P}^3$ is equal to $\binom{3+5}{5} = 56$. Every line imposes 6 conditions on forms of degree 5. Since Y consists of 9 lines, We expect it to impose $9 \cdot 6 = 54$ conditions, so the dimension of space of forms of degree 5 vanishing on Y is expected to be $56 - 54 = 2$ and we expect ϕ to be surjective. For Y' the number of expected imposed conditions is $60 > 56$, which means that the kernel of ϕ is trivial, but ϕ is not a bijection, it is only an injection.

The problem is that for $d = 5$ proving that for 9 general lines ϕ is surjective and for 10 general lines ϕ is injective is not sufficient to prove the Theorem. It can be understood by the fact that although the surjectivity of ϕ for 9 general lines proves the fact that 9 general lines have good postulation, for 10 general lines ϕ can be injective even if the 10-th general lines implies less independent conditions (e.g. only 3 independent conditions instead of 6).

4.1 Reduction of the statement of Theorem 4.1.

Since a single line L imposes $k+1$ conditions on forms of degree k in $\mathbb{P}^3$, an upper bound for r general lines is given by $\operatorname{codim} \ker \phi \leqslant r(k+1)$. Otherwise, this would imply that some line imposes more than $k+1$ conditions on forms of degree k.

Let us assume that $\mathbb{N} \ni r = \dfrac{\binom{3+k}{k}}{k+1}$. If such r exists, then ϕ for $Y = L_1 + \ldots + L_r$ is expected to be bijective as explained in Example 4.2. Since we have upper bounding for the codimension, we know that ϕ is injective, therefore proving that it is also a surjection is enough to prove the theorem, because it proves the surjection for $r' < r$ and it guarantees trivial kernel (injection) for $r' > r$.

Corollary 4.4. *Let $k \geqslant 0$, be any set integers. If there exists a natural number r such as $r = \dfrac{\binom{3+k}{k}}{k+1}$, To prove Theorem 4.1 for set k it suffices to prove that for Y consisting of r lines in the general position, i.e. $Y = L_1 + \cdots L_r$ the only form of degree k vanishing on Y is a zero polynomial.*

More generally, we can define $r = \left\lfloor \frac{\binom{3+k}{k}}{k+1} \right\rfloor$ and q as the remainder: $q = \binom{3+k}{k} - r(k+1)$. If the remainder $q \neq 0$, it means that there exists no r such as ϕ is bijection.

Proposition 4.5. *Let $k \geqslant 0$, be any set integers. We define r, q as before: $r = \lfloor \frac{\binom{3+k}{k}}{k+1} \rfloor$ and $q = \binom{3+k}{k} - r(k+1)$. Let Y be the set of r lines in the general position and q collinear points, i.e. $Y = L_1 + \cdots L_r + P_1 + \cdots P_q$. It suffices to prove that ϕ for such Y has trivial kernel to prove Theorem 4.1 for the set k.*

Proof. It is a generalization of Corollary 4.4, when $q \neq 0$. To conclude this, we use Lemma 2.14 which states that $q \leqslant k+1$ collinear points on $\mathbb{P}^3$, impose q independent conditions on the form of degree k. Using the upper bounding of $dimker(\phi)$ and the mentioned lemma, we can see that the trivial kernel of ϕ for such Y proves that it is bijective and therefore proves that for every $r' < r$ lines in general position, the codimension is expected and since the q points $P_1, \cdots, P_q$ are collinear, if we add a new line in general position but passing through $P_1, \cdots, P_q$, the kernel must remain trivial. □

Remark 4.6. The triviality of the kernel mentioned in Proposition 4.5 means that the dimension of the space of forms of degree d vanishing in r general lines and q collinear points is zero. Since the dimension function is upper semi-continuous, it suffices to prove the statement for a *particular* configuration of lines and points.

Chapter 5

Proof of the Main Theorem

In order to alleviate notation, the statement of Theorem 4.1 for a fixed $d \geqslant 0$ will be denoted by H_d. Using Proposition 4.5, we will prove Theorem 4.1 for any d by induction. First, we need to deal with the initial cases for $d \in \{0, \dots, 8\}$.

5.1 Initial cases

Case $H_0, r = 1,\ q = 0$**.** We have to show that the non-trivial form of degree zero does not vanish on a general line. For example, for a line L given by equations

$$\begin{cases} x = 0 \\ y + z = 0 \end{cases}$$

such form of degree 0 does not exist which proves the statement.

Case H_1**,** $r = 2, q = 0$. Two general lines in $\mathbb{P}^3$ are skew, so they are not contained in any plane.

Case H_2 $r = 3,\ q = 1$. We want to prove that there exists no quadric containing three general lines and one general point. We start by showing that there exists exactly one quadric containing three general lines.

We know that there exists at least one quadric containing three general lines because three lines may impose at most $3 \cdot (2 + 1) = 9$ conditions on the quadric and the dimension of all forms of degree 2 in $\mathbb{P}^3$ is equal to $\binom{3+2}{3} = 10 > 9$. By the semicontinuity of dimensions of cohomology spaces, it suffices to construct three lines such that the dimension of a space of quadrics containing those three lines is equal to 1.

Consider the lines L_1, L_2, L_3 given by equations

$$\begin{cases} x = & 0 \\ z = & 0 \end{cases}, \quad \begin{cases} y = & 0 \\ w = & 0 \end{cases}, \quad \begin{cases} x = & -y \\ z = & -w \end{cases}$$

respectively. It is easy to see that they can be parametrized in the the following way:

$$\begin{aligned} L_1 &= \{(0 : u : 0 : v) \mid (u : v) \in \mathbb{P}^1\} \\ L_2 &= \{(u : 0 : v : 0) \mid (u : v) \in \mathbb{P}^1\} \\ L_3 &= \{(-u : u : -v : v) \mid (u : v) \in \mathbb{P}^1\} \end{aligned}$$

Now consider the quadric Q given by an equation

$$Ax^2 + By^2 + Cz^2 + Dw^2 + Exy + Fxz + Gxw + Hyz + Iyw + Jzw = 0$$

Due to the fact that Q contains L_1 the equation of Q must be satisfied for every point from L_1 in particular the points $(0 : 1 : 0 : 0), (0 : 0 : 0 : 1), (0 : 1 : 0 : 1)$. Substituting these points into the equation of Q we obtain the following system of linear equations

$$\begin{cases} D = 0 \\ B = 0 \\ D + B + I = 0 \end{cases}.$$

Similarly taking advantage of the fact that Q contains L_2 and considering points $(1 : 0 : 0 : 0), (0 : 0 : 1 : 0), (1 : 0 : 1 : 0) \in L_2$, we obtain the system of equations

$$\begin{cases} A = 0 \\ C = 0 \\ A + C + F = 0 \end{cases}.$$

From those 6 equations combined we know that $A = B = C = D = F = I = 0$. Using that and considering points $(-1 : 1 : 0 : 0), (0 : 0 : -1 : 1), (-1 : 1 : -1 : 1) \in L_3$ we obtain the following equations

$$\begin{cases} E = 0 \\ J = 0 \\ E + J + G + H = 0 \end{cases}.$$

In the end we have $A = B = C = D = E = F = I = J = 0$ and $G = -H$ which gives us an equation of quadric Q in a form of

$$t \cdot xw - t \cdot yz = 0, \quad t \in \mathbb{C}.$$

This proves that the dimension of quadrics vanishing on three general lines in $\mathbb{P}^3$ is equal to 1.

Now, since there is only one quadric containing three general lines, we can ensure that by taking a point outside of the quadric the dimension of quadrics vanishing on three general lines and general point is equal to zero, meaning that the only such form is the zero form.

Case H_3 $r = 5, q = 0$. We would like to prove that the only form of degree 3 vanishing on five general lines $L_1, \ldots, L_5$ is the zero form. Working by contradiction, let us assume that there exists a non-trivial form F of degree 3 vanishing along the lines.

Consider the quadric Q determined by L_1, L_2, L_3. The lines L_4, L_5 intersect Q transversally, thus each in two points (see Figure 5.1). There are two cases: $Q \subset F$ or $Q \not\subset F$.

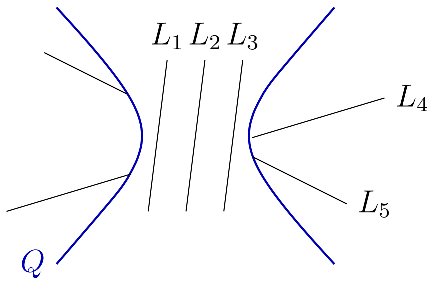


Figure 5.1: Arrangement of Q and $L_1, \ldots, L_5$.

1. Consider $Q \not\subset F$. Therefore, from Theorem 2.10 we know that the intersection $F \cap Q$ is a curve of degree 6. Consider the Segre isomorphism $S : \mathbb{P}^1 \times \mathbb{P}^1 \to Q$ and its inverse S^{-1}. By Theorem 3.6 we know that $S^{-1}(F \cap Q)$ is a curve $\mathfrak{C}$ of bidegree $(3, 3)$. Since $L_1 + \ldots + L_5 \subset F$, therefore $\mathfrak{C}$ contains the preimage of restrictions of $L_1, \ldots, L_5$ to Q. As L_1, L_2, L_3 are in the same ruling (wlog. vertical ruling) in Q from 3.8 their preimage are three non-intersecting lines $\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3$ in $\mathbb{P}^1 \times \mathbb{P}^1$ each of bidegree $(1, 0)$. The preimage of L_4, L_5 that intersects with Q in two general points is simply four general points $\mathfrak{p}_1, \ldots, \mathfrak{p}_4$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Since $\mathfrak{C}$ vanishes on the lines $\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3$, factoring these lines out, we obtain a form of bidegree $(0, 3)$ (hence 3 lines in the horizontal ruling) that must contain four general points $\mathfrak{p}_1, \ldots, \mathfrak{p}_4$, which is a contradiction.
2. Consider $Q \subset F$. Therefore, we can write $F = Q + \widetilde{F}$ where $\widetilde{F}$ is a form of degree $3 - 2 = 1$. Since L_4, L_5 are not contained in Q, it is required that $L_4 + L_5 \subset \widetilde{F}$ which contradicts H_1.

Case H_4 $r = 7,\ q = 0$. The goal is to show that for seven general lines, the only form of degree 4 vanishing on all seven lines is the zero polynomial. We will utilize similar techniques to the case H_3 with small twists. Let denote the lines $L_1, \ldots, L_7$ and let L_1, L_2, L_3 be lines in the vertical ruling of a smooth quadric Q.

We specialize L_4 and L_5 to intersect at a point and moreover we assume that this point lies on Q. We do the same for the pair L_6 and L_7. Of course we assume that the intersection points of each pair are different (see Figure 5.2).

Let us now assume by contradiction that there exists a non-trivial form of degree 4 denoted by F vanishing along all lines $L_1, \ldots, L_7$. Then again there are two cases: $Q \subset F$ or $Q \not\subset F$.

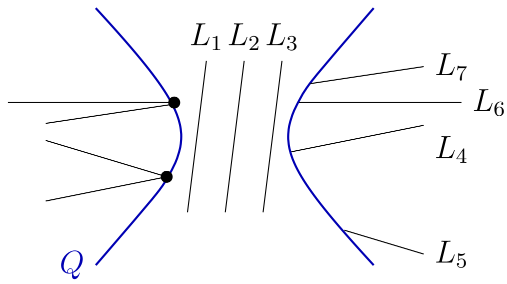


Figure 5.2: Arrangement of Q and $L_1, \ldots, L_7$.

1 Consider first the case $Q \not\subset F$. Then the restriction $F|_Q$ is a curve of degree $2 \cdot 4 = 8$. Again considering the Segre isomorphism $S : \mathbb{P}^1 \times \mathbb{P}^1 \to Q$ and it's inversion S^{-1} we know that $\mathfrak{C} = S^{-1}(F|_Q)$ is a bihomogeneous curve of bidegree $(4, 4)$ vanishing on all traces of $L_1, \ldots, L_7$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Since L_1, L_2, L_3 are contained in Q their image in $\mathbb{P}^1 \times \mathbb{P}^1$ are three lines $\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3$ in the vertical ruling (they are not intersecting). Therefore $\mathrm{bideg}(\mathfrak{C} - (\mathfrak{L}_1 + \mathfrak{L}_2 + \mathfrak{L}_3)) = (4, 4) - 3 \cdot (1, 0) = (1, 4)$.

Now the traces of L_4, L_5, L_6, L_7 are two double points (points with multiplicity two coming from intersections of L_4, L_5 and L_6, L_7 respectively) and four points with multiplicity one. Suppose there exist a bihomogeneous polynomial g of bidegree $(1, 4)$ vanishing at all those points. Consider now a line of bidegree $(0, 1)$ (a horizontal line) passing through one of the points with multiplicity 2. We know that g has multiplicity two at this point, thus g has to contain the whole line because otherwise it would contradict Bézout's Theorem for $\mathbb{P}^1 \times \mathbb{P}^1$ 3.5, since curves of bidegree $(1, 4)$ and $(0, 1)$ intersect either at exactly 1 point counted with the multiplicity or the second curve is a component of the first curve. The same argument can be applied to prove that the polynomial g vanishes aling line of bidegree $(0, 1)$ containing the second point with multiplicity two.

Reducing g by those 2 lines we are left with a polynomial of bidegree $(1, 4) - (0, 1) - (0, 1) = (1, 2)$ containing 6 general points with multiplicity one. By Lemma 3.4 we know that the dimension of a space of polynomials of bidegree $(1, 2)$ is equal to $(1 + 1) \cdot (2 + 1) = 6$ so the only polynomial of such degree containing 6 general points is the zero polynomial, which is a contradiction.

2. In the case $Q \subset F$ we have $F = Q + \widetilde{F}$, where $\widetilde{F}$ is a form of degree $4 - 2 = 2$ in $\mathbb{P}^3$. Since L_4, L_5, L_6, L_7 are not contained by Q they must lie in $\widetilde{F}$. If $L_1, \ldots, L_4$ were lines in general position we could say that it is a contradiction with H_2 but remember we actually specialized L_4, L_5 to intersect and L_6, L_7 to intersect as well. So the argument does not apply directly.

If the quadric $\widetilde{F}$ contains all those lines, we can again use the Segre isomorphism to see how traces of those lines look in $\mathbb{P}^1 \times \mathbb{P}^1$. We denote images of $L_4, \ldots, L_7$ by $\mathfrak{L}_4, \ldots, \mathfrak{L}_7$ respectively. Since L_4 and L_5 intersected thus their images also do, which means $\mathfrak{L}_4, \mathfrak{L}_5$ are lines from different ruling on $\mathbb{P}^1 \times \mathbb{P}^1$. Without loss of generality we can write that $\mathfrak{L}_4$ is of bidegree $(1, 0)$ and $\mathfrak{L}_5$ is of bidegree $(0, 1)$. Exactly same logic allows us to write that $\mathfrak{L}_6$ is of bidegree $(1, 0)$ and $\mathfrak{L}_7$ is of bidegree $(0, 1)$. Since every two lines from different rulings intersect in $\mathbb{P}^1 \times \mathbb{P}^1$, this implies that $\mathfrak{L}_4$ and $\mathfrak{L}_7$ intersects and $\mathfrak{L}_5$ intersects with $\mathfrak{L}_6$. In the consequence L_4 and L_7 have to intersect and L_5 and L_7 have to intersect which is not true in general, therefore it is a contradiction.

Case H_5 $r = 9,\ q = 2$. We want to show that there exists no form of degree 5 that contains nine general lines and two points. Working by contradiction, let us assume that there exists a non-trivial form F of degree 5 vanishing along the lines $L_1, \ldots, L_9$ and points P_1, P_2. We specialize $L_1, \ldots, L_4$ to lie in the vertical ruling of the smooth quadric Q and we specialize the points P_1, P_2 to be contained in the same quadric Q. Lastly, the lines $L_5, \ldots, L_9$ are general, so they intersect Q transversally, each at two points (see Figure 5.3). There are two cases to consider. Either $Q \subset F$ or $Q \not\subset F$.

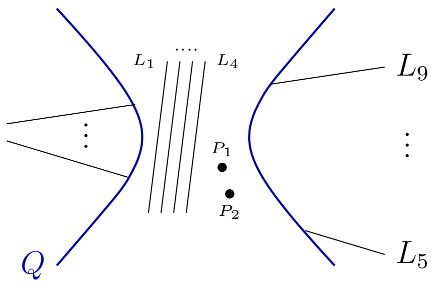


Figure 5.3: Arrangement of Q, $L_1, \ldots, L_9$ and P_1, P_2.

1 Consider the case $Q \not\subset F$. Since $L_5, \ldots L_9$ intersected Q transversally, the traces of $L_5, \ldots L_9$ are ten general points on $\mathbb{P}^1 \times \mathbb{P}^1$. Adding two general points $S^{-1}(P_1), S^{-1}(P_2)$ we get twelve general points in $\mathbb{P}^1 \times \mathbb{P}^1$. Since by Lemma 3.4 we know that the dimension of the forms of the bidegree $(1, 5)$ is $2 * 6 = 12$, we know that the only form of the bidegree $(1, 5)$ that vanishes in those twelve points is the zero form; hence, we get a contradiction.

2 Consider the case $Q \subset F$. Then we can write $F = Q + \widetilde{F}$, where $\widetilde{F}$ is a form of degree $5 - 2 = 3$ in $\mathbb{P}^3$. Since $L_5, \ldots, L_9$ are not contained in Q, they must be contained in $\widetilde{F}$. The form of degree 3 vanishing along five general lines is in contradiction with H_3, which ends the proof.

Case H_6 $r = 12,\ q = 0$. We want to prove that there exists no form of degree 6 vanishing along twelve general lines. By contradiction, let us assume that there is a form F of degree 6 containing the lines $L_1, \ldots, L_{12}$. We specialize $L_1, \ldots L_5$ to lie on the vertical ruling of a smooth quadric Q, and the lines $L_6, \ldots L_{12}$ are general, which means that they intersect Q transversally, each at two distinct points (see Figure 5.4). There are two cases to consider, either $Q \subset F$ or $Q \not\subset F$.

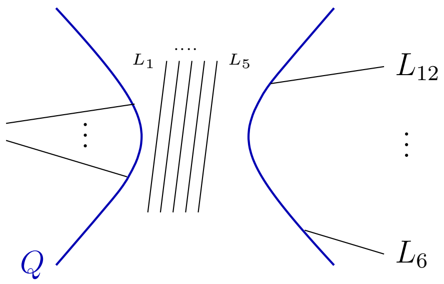


Figure 5.4: Arrangement of Q, and $L_1, \ldots, L_{12}$.

1 Consider the case $Q \not\subset F$. Then, from Bézout's Theorem 2.10 and 3.6 we know that $\mathfrak{C} = S^{-1}(F|_Q)$, where S denotes the Segre isomorphism, is a curve of bidegree $(6, 6)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. If we denote $\mathfrak{L}_i = S^{-1}(L_i)$ for $i = 1 \ldots, 5$, since $L_1, \ldots, L_5 \subset Q$, we know that $\mathfrak{L}_1, \ldots, \mathfrak{L}_5 \subset \mathfrak{C}$. Note that from 3.8 $\mathfrak{L}_i$ is a line of bidegree $(1, 0)$ for $i = 1, \ldots, 5$. Therefore we can write $\deg(\mathfrak{C} - (\mathfrak{L}_1 + \ldots + \mathfrak{L}_5) = (6, 6) - 5 \cdot (1, 0) = (1, 6)$. This means that the residue curve of bidegree $(1, 6)$ must contain all traces of $L_6, \ldots, L_{12}$ in $\mathbb{P}_1 \times \mathbb{P}_1$. The preimage of $L_6, \ldots, L_{12}$ is 14 general points (originating from the intersections of $L_6, \ldots, L_{12}$ with Q). By Lemma 3.4 the dimension of all forms of bidegree $(1, 6)$ is equal to $2 \cdot 7 = 14$, therefore there exists no non-trivial form of such bidegree containing 14 general points which results in a contradiction.

2 Consider the case $Q \subset F$. Then, we may write $F = Q + \widetilde{F}$ for $\widetilde{F}$ being a form of degree $6 - 2 = 4$. Since $L_6, \ldots, L_{12} \not\subset Q$, we know that $L_1, \ldots, L_{12} \subset \widetilde{F}$. This is a direct contradiction with H_4, which ends the proof.

Case H_7 $r = 15$, $q = 0$.. We want to show that there exists no form of degree 7 vanishing along 15 general lines. Working by contradiction, let us assume that there is a form F is of degree 7 vanishing along general lines $L_1, \ldots L_{15}$. We begin by specializing lines $L_1, \ldots, L_5$ to the vertical ruling of a smooth quadric Q. Additionally, we specialize lines L_6 and L_7 to intersect exactly at a point on Q. We do the same with three next pairs of lines: L_8 with L_9, L_{10} with L_{11} and L_{12} with L_{13}. That way, we end up with five lines fully contained by Q, four pairs of lines intersecting at a point in Q and two general lines. We have to consider two cases, either $Q \subset F$ or $Q \not\subset F$.

1 Consider first $Q \not\subset F$. Considering the Segre isomorphism $S : \mathbb{P}^1 \times \mathbb{P}^1$, by 2.10, we know that the $F|_Q$ is a curve of degree 14 and by 3.6 we know that $\mathfrak{C} = S^{-1}(F|_Q)$ is a curve of bidegree $(7, 7)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Let us denote $\mathfrak{L}_i = S^{-1}(L_i)$ for $i = 1, \ldots, 5$. By 3.8, we know that for $i = 1, \ldots, 5$ the preimage $\mathfrak{L}_i$ is a line of bidegree $(1, 0)$. Since $L_1 \ldots, L_{15} \subset F$, we know that $\mathfrak{C}$ vanishes along all preimages of traces of L_i. Consider the residue curve obtained by factoring $\mathfrak{L}_1, \ldots, \mathfrak{L}_5$ out of $\mathfrak{C}$. Since we know that $\deg(\mathfrak{C} - (\mathfrak{L}_1 + \ldots + \mathfrak{L}_5)) = (7, 7) - 5 \cdot (1, 0) = (2, 7)$, the residue curve is of bidegree $(2, 7)$ and contains

preimages of all traces of $L_6, \dots, L_{15}$. Consider lines $L_6, \dots, L_{13}$. Those lines consists of four pairs of lines intersecting exactly at a point in Q. Additionally, each of those 8 lines intersects Q in one more point. Therefore we have 4 points with multiplicity two and 8 points with multiplicity one. Taking into account the two last general lines L_{14}, L_{15} which intersect at 4 more points with Q we have 4 points with multiplicity two and $8 + 4 = 12$ points with multiplicity one. All those points have to be contained by a form of bidegree $(2, 7)$. By 3.12 we know that such arrangement of points with multiplicity two and one imposes independent conditions in $\mathbb{P}^1 \times \mathbb{P}^1$, which means that every double point imposes three conditions on forms of any bidegree, and points general points with multiplicity one impose one condition each. In total, we have $3 \cdot 4 + 12 = 24$ conditions. Due to the fact, that the dimension of a space of forms of bidegree $(2, 7)$ by 3.4 is equal to $3 \cdot 8 = 24$, we get a contradiction.

2 Consider $Q \subset F$. This case is especially interesting, because proving contradiction is not as straightforward as it was in earlier examples. We can write $F = Q + \widetilde{F}$, where $\widetilde{F}$ is a form of degree $7-2 = 5$ containing four pairs of intersecting lines $L_6, \dots, L_{13}$ and two general lines L_{14}, L_{15}. Now let us additionally specialize those lines. We choose lines $L_6, L_8, L_{10}, L_{12}, L_{14}$ to lie on a vertical ruling of some smooth quadric Q'. Notice that, we now have specialized one line from each intersecting pair of lines and one of the remaining general lines. Consider the Segre isomorphism S acting on Q'. We have two consider now two cases. Either $Q' \subset \widetilde{F}$ or $Q \not\subset \widetilde{F}$.

a) Consider the case $Q \not\subset \widetilde{F}$. From the Bezout's theorem 2.10 and Theorem 3.6, we know that $\mathfrak{S} = S^{-1}(\widetilde{F}\Big|_{Q'})$ is a curve of bidegree $(5, 5)$. Since $L_6, L_8, L_{10}, L_{12}, L_{14} \subset Q'$ by Theorem 3.8 we know that $\mathfrak{L}_i = S^{-1}(L_i), i = 6, 8, 10, 12, 14$ are lines of bidegree $(1, 0)$. Consider the residue curve obtained by factoring $\mathfrak{L}_6, \mathfrak{L}_8, \mathfrak{L}_{10}, \mathfrak{L}_{12}, \mathfrak{L}_{14}$ out from the $\mathfrak{C}$. We may write $\deg(\mathfrak{S} - \sum_{k=3}^{7} \mathfrak{L}_{2k}) = (5, 5) - 5 \cdot (1, 0) = (0, 5)$. This curve of bidegree $(0, 5)$ have two contain all the preimages of traces of $L_7, L_9, L_{11}, L_{13}, L_{15}$ on Q'. The L_{15} is a general line therefore it intersects with Q' at two distinct points. Lines L_7, L_9, L_{11}, L_{13} also intersects with Q' in two points, however one of this points is already contained by Q', because it comes from the intersection with paired line L_6, L_8, L_{10}, L_{12} respectfully. In the end we have $2 + 4 \cdot 1 = 6$ general points contained by a curve of bidegree $(0, 5)$. Since the dimension of a space of forms of bidegree $(0, 5)$ is equal to 6, such form does not exist and it is a contradiction.

b) Consider the case $Q' \subset \widetilde{F}$. Then, we can write $\widetilde{F} = Q' + \widetilde{\widetilde{F}}$, where $\widetilde{\widetilde{F}}$ is a form of degree 3 containing $L_7, L_9, L_{11}, L_{13}, L_{15}$. It is a form of degree 3 containing 5 general lines, which is a direct contradiction with H_3.

The case of H_7 is clearly different and more complex than previously considered cases. For H_4 the complexity was hidden under small number of lines, which allowed us to use some other tricks. However, in the generalized proof the cases where $d = 3k + 1$ remain as complex as H_7. To handle them, we will always rely on the techniques presented in this proof.

Case H_8 $r = 18,\ q = 3$. We want to show that there exists no form of degree 8 vanishing in 18 general lines and 3 collinear points. Working by contradiction, let us assume that the form F vanishes on the lines $L_1, \dots, L_{18}$ and the collinear points P_1, P_2, P_3. We specialize lines $L_1, \dots, L_6$ to lie in a vertical ruling of a smooth quadric Q. Additionally, we specialize the points P_1, P_2, P_3 not only to be collinear but also to be contained in the same quadric Q. We have to consider two cases, either $Q \subset F$ or $Q \not\subset F$.

1 Consider $Q \not\subset F$. Considering the Segre isomorphism $S : \mathbb{P}^1 \times \mathbb{P}^1$, by 2.10, we know that $F|_Q$ is a curve of degree 16 and by 3.6, we know that $\mathfrak{C} = S^{-1}(F|_Q)$ is a curve of bidegree $(8, 8)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Consider $\mathfrak{L}_i = S^{-1}(L_i)$ for $i = 1, \dots, 6$. By 3.8, we know that $\mathfrak{L}_i$ is a line on the vertical ruling i.e. a curve of bidegree $(1, 0)$. Consider the residue curve obtained by factoring $\mathfrak{L}_1, \dots, \mathfrak{L}_6$ out of $\mathfrak{C}$. Since we know that $\deg(\mathfrak{C} - (\mathfrak{L}_1 + \ldots + \mathfrak{L}_6)) = (8, 8) - 6 \cdot (1, 0) = (2, 8)$, the residue curve is of bidegree $(2, 8)$ and contains preimages of all traces of $L_7, \dots, L_{18}$ and traces of P_1, P_2, P_3 left on $\mathbb{P}^1 \times \mathbb{P}^1$. Since each line $L_7, \dots, L_{18}$ intersected with Q at two distinct points, the $S^{-1}((L_7 \cup \ldots \cup L_{18}) \cap Q)$ are 24 general points in $\mathbb{P}^1 \times \mathbb{P}^1$. Since P_1, P_2, P_3 where chosen to lie in Q their preimages are simply 3 collinear points on $\mathbb{P}^1 \times \mathbb{P}^1$. The residue curve vanishes on 24 general points and 3 collinear points. By Theorem 3.11 we know that 3 collinear points on $\mathbb{P}^1 \times \mathbb{P}^1$ impose independent conditions on forms of bidegree $(2, 8)$. Thus, in total, we have 27 independent conditions imposed on a residue curve of bidegree (2.8). Since the dimension of all forms of bidegree $(2, 8)$ is 27, the only form of such a bidegree satisfying all conditions is the zero polynomial, and hence it is a contradiction.

2 Consider $Q \subset F$. Then, we can write $F = Q + \widetilde{F}$, where $\widetilde{F}$ is a form of degree $8 - 2 = 6$ vanishing along $L_7, \dots, L_{18}$. Since $L_7, \dots, L_{18}$ are general lines, it is a contradiction with H_6.

5.2 Proof by induction for all $d \geqslant 9$

The eight initial cases of H_d should be enough to grasp a concept of how the general proof is going to work. We simply consider three distinct cases, based on the reminder of d modulo 3. For each case, we construct a slightly different arrangement of lines, proving the theorem. Since the initial cases are already proven, we perform the proof by induction.

Case $d = 3k,\ k \in \mathbb{N}_{\geqslant 2}$. $r = \frac{1}{2}(k+1)(3k+2),\ q = 0$. First, we briefly explain how r was calculated. In the later cases of proofs, we omit this part. The dimension of all forms of degree $3k$ in $\mathbb{P}^3$ is equal to $\binom{3k+3}{3} = \frac{1}{2}(k+1)(3k+2)(3k+1)$. Any single line imposes $3k+1$ independent conditions on forms of degree $3k$, therefore, we consider $r = \frac{\frac{1}{2}(k+1)(3k+2)(3k+1)}{3k+1} = \frac{1}{2}(k+1)(3k+2)$. It is easy to observe that r is an integer for all k. We want to prove that there exists no non-trivial form of degree $3k$ that contains r general lines. Working by contradiction, we assume that a nonzero form F of degree $3k$ vanishes along Y, where $Y = L_1 + \ldots + L_r$ is the union of r lines. We specialize Y to be of the form $Y = Y' + Y''$, where $Y' = L_1 + L_{2k+1}$ is the set of $2k+1$ lines that lie in the vertical ruling of a smooth quadric Q. Note that $2k+1 < r$, so the construction is always possible. Y'' is the set of the remaining s lines, i.e. $s = r - (2k+1) = \frac{1}{2}(3k^2+5k+2) - (2k+1) = \frac{1}{2}(3k^2+k) = \frac{1}{2}k(3k+1)$. Each line of Y'' intersects with Q at two distinct points transversally.

We now consider two cases, $Q \subset F$ or $Q \not\subset F$.

1 Consider the case $Q \not\subset F$. By Bézout's theorem 2.10 we know that $F|_Q$ is a curve on Q of degree $2 \cdot 3k = 6k$. By 3.6, we know that $\mathfrak{C} = S^{-1}(F|_Q)$ is a form of bidegree $(3k, 3k)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, where $S : \mathbb{P}^1 \times \mathbb{P}^1 \to Q$ is the Segre isomorphism. Let us denote $\mathfrak{Y}' = S^{-1}(Y'|_Q)$ and $\mathfrak{Y}'' = S^{-1}(Y''|_Q)$. We can write $Y \subset F \Rightarrow S^{-1}(Y|_Q) \subset \mathfrak{C} \Leftrightarrow \mathfrak{Y}' \cup \mathfrak{Y}'' \subset \mathfrak{C}$. Since Y' is the union of $2k+1$ lines in the vertical ruling on Q, we know that $\mathfrak{Y}'$ is the union of $2k+1$ lines in the vertical ruling on $\mathbb{P}^1 \times \mathbb{P}^1$. Consider the residue curve $\mathfrak{R} = \mathfrak{C} - \mathfrak{Y}$. We know that $\deg(\mathfrak{R}) = (3k, 3k) - (2k+1) \cdot (1, 0) = (k-1, 3k)$. Note that since each line of the s lines of Y'' intersected with Q at two points, $\mathfrak{Y}''$ is a union of the $2 \cdot s = 2 \cdot \frac{1}{2}k(3k+1) = k(3k+1)$ points. We know that $\mathfrak{Y}'' \subset \mathfrak{R}$. Thus, we have a form of bidegree $(k-1, 3k)$ vanishing in $k(3k+1)$ points. Since the dimension of all forms of bidegree $(k-1, 3k)$ is equal to $k(3k+1)$, the only form of such a bidegree vanishing in $k(3k+1)$ general points is zero polynomial and that is a contradiction.

2 Consider the case $Q \subset F$. Then we may write $F = Q + \widetilde{F}$, where $\widetilde{F}$ is the form of degree $3k-2$ vanishing along Y''. Notice that H_{d-2} states that there exists no non-trivial form of degree $3k-2$ that vanishes along $\frac{1}{2}k(3k+1)$ general lines[1]. Since Y'' consists exactly of the $\frac{1}{2}k(3k+1)$ general lines, we get a contradiction with H_{d-2}, ending the proof.

Case $d = 3k+1,\ k \in \mathbb{N}_{\geqslant 2}$**.** $r = \frac{1}{2}(k+1)(3k+4),\ q = 0$. We want to show that there exists no non-trivial form of degree $3k+1$ vanishing along r lines in general position. Working by contradiction let us assume that F is a non-trivial form of such degree vanishing in $Y = L_1 + \ldots + L_r$, where L_i is a line for all $i = 1, \ldots, r$. We write Y in the form of $Y = Y' + C_1 + \ldots C_{2k} + Y''$, where $Y' = L_1 + \ldots L_{2k+1}$ is a union of $2k+1 < r$ lines, $C_i = L_{2(k+i)} + L_{2(k+i)+1}$ is a pair of lines for $i = 1, \ldots, 2k$ and Y'' is a union of remaining $r - (2k+1) - 2\cdot(2k) = \frac{1}{2}(3k^2+7k+4) - 6k - 1 = \frac{1}{2}(3k^2-5k+2) > 0$ lines. We specialize Y' to lie in the vertical ruling of some smooth quadric Q.

[1]To calculate this, look at the r for case $d = 3k+1$, and substitute $k-1$ in place of k.

Additionally, we specialize the pairs C_i for all $i = 1, \dots, 2k$ in the following way: two lines from the same pair intersect with each other exactly at the point in quadric Q. The remaining lines in Y'' are in general position. We have two consider two cases, $Q \subset F$ and $Q \not\subset F$.

1 Consider the case $Q \not\subset F$. Then by Bezout's theorem 2.10 and Theorem 3.6 we know that $\mathfrak{C} = S^{-1}(F|_Q)$, where S denotes the Segre isomorphism, is a form of bidegree $(3k+1, 3k+1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ which vanishes along all preimages of traces of $L_1, \dots, L_r$. Since $Y' \subset Q$ we may consider $\mathfrak{Y}' = S^{-1}(Y')$. By 3.8, we know that $\mathfrak{Y}'$ is a union of $2k+1$ lines of bidegree $(1, 0)$. Additionally let us denote $\mathfrak{Y}'' = S^{-1}(Y''|_Q)$ and $\mathfrak{P}_i = S^{-1}(C_i|_Q)$. Since Y'' consisted of $\frac{1}{2}(3k^2 - 5k + 2)$ general lines, each intersecting Q at two points, $\mathfrak{Y}''$ is a union of $3k^2 - 5k + 2$ points on $\mathbb{P}^1 \times \mathbb{P}^1$. Each C_i intersects with Q at three points. Two standard points with multiplicity one, and one additional point with multiplicity two coming from the intersection of lines consisting C_i. Therefore every $\mathfrak{P}_i$ is a union of three points, one with multiplicity two and two with multiplicity one in $\mathbb{P}^1 \times \mathbb{P}^1$. Consider the residue curve obtained by factoring $\mathfrak{Y}'$ from $\mathfrak{C}$. We may write $\deg(\mathfrak{C} - \mathfrak{Y}') = (3k+1, 3k+1) - (2k+1) \cdot (1, 0) = (k, 3k+1)$. The residue curve contains $\mathfrak{Y}'', \mathfrak{P}_1, \dots, \mathfrak{P}_{2k}$, which are in total $3k^2 - 5k + 2 - 2k + 2 + 2 \cdot (2k) = 3k^2 - k + 2$ points with multiplicity one and $2k$ points with multiplicity two. By Theorem 3.12 we know that such arrangements of points imposes independent conditions on forms of such bidegree, meaning that each points with multiplicity one imposes one condition and each point with multiplicity two imposes 3 conditions. Thus, in total $\mathfrak{Y}'' + \sum_{i=1}^{2k} \mathfrak{P}_i$ imposes $3k^2 - k + 2 + 2k \cdot 3 = 3k^2 + 5k + 2$ independent conditions. Since the dimension of all forms of bidegree $(k, 3k+1)$ is equal to $(k+1)(3k+2) = 3k^2 + 5k + 2$, the form $\mathfrak{C}$ must be the zero form, which is a contradiction with $Q \not\subset F$.

2 Consider the case $Q \subset F$. Then, we can write $F = Q + \widetilde{F}$, where $\widetilde{F}$ is the form of degree $3k - 1$ vanishing in $C_1, \dots, C_{2k}, Y''$. To finish the proof, we must specialize our arrangements even more. Let us denote $R = \sum_{i=1}^{2k} L_{2(k+i)}$ and $R = \sum_{i=1}^{2k} L_{2(k+i)+1}$. Notice that for each line pair C_i one of lines is contained in R and the second one is contained in R'. We also define $Y''' = Y'' - L_r$. Now we specialize R and L_r to lie in a vertical ruling of some smooth quadric Q'. We have to consider yet another two cases. $Q' \subset \widetilde{F}$ and $Q' \not\subset \widetilde{F}$.

   a) Consider the case $Q' \not\subset \widetilde{F}$. Considering S as Segre isomorphism acting on Q', from the Bezout's theorem 2.10 and Theorem 3.6 we know that $\mathfrak{F} = S^{-1}(\widetilde{F}\Big|_{Q'})$ is a curve of bidegree $(3k-1, 3k-1)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ that contains all of preimages of the traces of R, L_r, R', Y''' in $\mathbb{P}^1 \times \mathbb{P}^1$. Since $R + L_r$ are $2k+1$ lines contained in vertical ruling of Q', we know that $\mathfrak{R} = S^{-1}(R)$ and $\mathfrak{L}_r = S^{-1}(L_r)$ combined are $2k+1$ lines of bidegree $(1, 0)$ contained in $\mathfrak{F}$. Therefore we can consider the residue curve $\mathfrak{F} - (\mathfrak{R} + \mathfrak{L}_r)$ and we can write $\deg(\mathfrak{F} - (\mathfrak{R} + \mathfrak{L}_r)) = (3k-1, 3k-1) - (2k+1) \cdot (1, 0) = (k-2, 3k-1)$.

The residue curve must also contain the preimage $\mathfrak{R}' = S^{-1}(R'|_{Q'})$ and the preimage $\mathfrak{Y}''' = S^{-1}(Y'''|_{Q'})$. Since Y''' is union of $\frac{1}{2}(3k^2-5k+2)-1$ general lines intersecting with Q at two distinct points each, we know that $\mathfrak{Y}'''$ are $3k^2-5k$ points with multiplicity one in general position in $\mathbb{P}^1 \times \mathbb{P}^1$. Now, R is a union of $2k$ lines, each intersecting with Q at two points, but for each line from R one point of intersection with Q is a point of intersection with a paired line from R, which is already contained by $\mathfrak{F}$. Thus $\mathfrak{R}'$ is a union of $2k$ points in $\mathbb{P}^1 \times \mathbb{P}^1$. In total we have $3k^2-5k+2k = 3k^2-3k$ points contained by residue curve of a bidegree $(k-2, 3k-1)$. By 3.4 the dimension of all forms of bidegree $(k-2, 3k-1)$ is equal to $(k-1)(3k) = 3k^2-3k$. Thus, a form of bidegree $(k-2, 3k-1)$ containing $3k^2-3k$ general points must be zero form, which is a contradiction with $Q' \not\subset \widetilde{F}$.

b) Consider the case $Q' \subset \widetilde{F}$. Then, we can write $\widetilde{F} = Q' + \widetilde{\widetilde{F}}$, where $\widetilde{\widetilde{F}}$ is a form of degree $3k-3$ vanishing in Y''' and R'. The $Y''' + R'$ is a union of $\frac{1}{2}(3k^2-5k)+2k = \frac{1}{2}(3k^2-k) = \frac{1}{2}(k+1)(3k-1)$ lines in general position. The form of degree $3k-3$ vanishing along $\frac{1}{2}(k+1)(3k-1)$ lines is a direct contradiction with $H_{3k-3} = H_{d-4}$.

Case $d = 3k+2,\ k \in \mathbb{N}_{\geqslant 2}$**.** $r = \frac{1}{2}(k+1)(3k+6),\ q = k+1$. The proof is very similar to the case $d = 3k$, but we have to take into account additional points. We want to show that there exists no non-trivial form of degree $3k+2$ vanishing along r lines in general position and q collinear points. By contradiction, we assume that F is a non-trivial form of degree $3k+2$ vanishing along $Y = L_1 + \ldots + L_r + P_1 + \ldots + P_q$, where $L_1, \ldots, L_r$ are lines and $P_1, \ldots, P_q$ are collinear points. We write Y in the form of $Y = Y' + Y'' + P$ where $Y' = L_1 + \ldots + L_{2k+2}$ is a union of $2k+2 < r$ lines, Y'' is the set of remaining $s = r-(2k+2) = \frac{1}{2}(3k^2+9k+6)-(2k+2) = \frac{1}{2}(3k^2+5k+2)$ lines and $P = P_1 + \ldots + P_q$ is a union of q collinear points. We specialize Y' to lie in the vertical ruling of a smooth quadric Q and P to be collinear points in the same quadric Q, while Y'' remains general, intersecting Q at $2s$ distinct points. We consider two cases, $Q \subset F$ or $Q \not\subset F$.

1 Consider the case $Q \not\subset F$. By Bézout's theorem 2.10 and by Theorem 3.6 we know that $\mathfrak{C} = S^{-1}(F|_Q)$ is a form of bidegree $(3k+2, 3k+2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Since $Y \subset F$, we know that $S^{-1}(Y|_Q) \subset \mathfrak{C}$. Denote $\mathfrak{Y}' = S^{-1}(Y'|_Q)$, $\mathfrak{Y}'' = S^{-1}(Y''|_Q)$ and $\mathfrak{P} = S^{-1}(P|_Q)$. Then we can write $\mathfrak{Y}' \cup \mathfrak{Y}'' \cup \mathfrak{P} \subset \mathfrak{C}$. Since Y' is a union of $2k+2$ lines contained in the vertical ruling of the quadric Q, by 3.8 we know that $\mathfrak{Y}'$ is also a union of $2k+2$ lines of bidegree $(1,0)$ each. Factoring these lines out of $\mathfrak{C}$ we obtain the residue curve $\mathfrak{R} = \mathfrak{C} - \mathfrak{Y}'$ and $\deg(\mathfrak{R}) = (3k+2, 3k+2) - (2k+2)\cdot(1,0) = (k, 3k+2)$. Since Y'' intersects with Q at $2s = 3k^2+5k+2$ general points, therefore $\mathfrak{Y}''$ is the set of $3k^2+5k+2$ general points in $\mathbb{P}^1 \times \mathbb{P}^1$ and since $P \subset Q$, we know that $\mathfrak{P}$ is the set of $q = k+1$ collinear points in $\mathbb{P}^1 \times \mathbb{P}^1$. We know that $\mathfrak{Y}'' \cup \mathfrak{P} \subset \mathfrak{R}$, which means that $\mathfrak{R}$ is

the form of bidegree $(k, 3k+2)$ vanishing in $3k^2+5k+2$ general points and $k+1$ collinear points. By Theorem 3.11, we know that $k+1$ collinear points in $\mathbb{P}^1$ impose independent conditions on the forms of bidegree $(k, 3k+2)$, therefore vanishing in $\mathfrak{Y}'' \cup \mathfrak{P}$ imposes $3k^2+5k+2+k+1 = 3k^2+6k+3 = (k+1)(3k+3)$ independent conditions. The dimension of all forms of bidegree $(k, 3k+2)$ is equal to $(k+1)(3k+3)$ therefore the only form of such bidegree vanishing on $\mathfrak{Y}'' \cup \mathfrak{P}$ is a zero polynomial, which is a contradiction.

2 Consider the case $Q \subset F$. Then, we can write $F = Q + \widetilde{F}$ where $\widetilde{F}$ is a form of degree $3k+2-2 = 3k$ vanishing along the lines contained in Y''. Notice that the statement of H_{d-2} is that there exists no non-trivial form of degree $d-2 = 3k$ that vanishes along the $\frac{1}{2}(k+1)(3k+2)$ lines. Since Y'' consists of $s = \frac{1}{2}(k+1)(3k+2)$, we get a contradiction that ends the proof.

Summary

In this thesis we provide a detailed proof of a result established in the 80's by Hartshorne and Hirschowitz to the effect that general lines in $\mathbb{P}^3$ (in fact in the projective space of any dimension) behave in the expected way with respect to imposing conditions on linear forms of arbitrary degree.

This is in strong contrast to the situation when no reduced structures are allowed, see [2] for the case of one fat line and [6] for some interesting cases with multiple fat lines.

Our approach is based on the well-established specialization and degeneration techniques going back to Castelnuovo. These techniques require tools in algebraic geometry and commutative algebra going far beyond the scope of regular classes at the bachelor level.

Streszczenie

W niniejszej pracy przedstawiamy szczegółowy dowód wyniku ustalonego w latach 80-tych przez Hartshorne'a i Hirschowitza, zgodnie z którym ogólne proste w $\mathbb{P}^3$ (a w istocie — w przestrzeni rzutowej dowolnego wymiaru) zachowują się zgodnie z oczekiwaniami, jeśli chodzi o narzucanie warunków na formy jednorodne dowolnego stopnia.

Stanowi to wyraźny kontrast wobec sytuacji, gdy dopuszcza się struktury niezredukowane: zob. [2] dla przypadku jednej grubej prostej oraz [6] dla kilku interesujących przypadków z wieloma grubymi prostymi.

Nasze podejście opiera się na dobrze ugruntowanych technikach specjalizacji i degeneracji sięgających czasów Castelnuovo. Techniki te wymagają narzędzi z geometrii algebraicznej i algebry przemiennej, które znacznie wykraczają poza zakres typowych zajęć na poziomie licencjackim.

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